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STABILIZATION OF COLLINEAR LIBRATION POINTS IN THE EARTH–MOON SYSTEM†

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The translational-rotational motion of an orbital station in the Earth–Moon system is investigated. The orbital station is regarded as a body of variable composition with a solid shell and a low-thrust jet engine placed on it, having constant autonomous orientation in a system of coordinates rotating with the Moon. It is shown that, by means of a reaction acceleration of small and constant modulus, one can stabilize both the new libration points themselves and the positions of relative equilibrium of the orbital station. Each value of the reaction acceleration, depending on its orientation, corresponds to a whole family of libration points, surrounding the classical collinear point, but only some of them can be stable. It is shown that, when the ellipticity of the Moon's orbit is taken into account, periodic translational-rotational motions of the orbital station in the neighbourhood of these points can occur with a period equal to the period of rotation of the Moon.

In previous investigations the problem of the stability of the positions of relative equilibrium of an orbital station in the neighbourhood of the collinear libration points of the Earth–Moon system has usually been solved in a restricted formulation: the orbital station was either assumed to be a point mass [1–4], or it was assumed that the location of the centre of mass of the orbital station (regarded as a solid or a gyrostat) at the libration point could be ensured a priori by special control forces [5], the nature of which and the form of the control were not discussed.

Consider the motion of an orbital station in the gravity field of the Earth and the Moon, assuming the latter to be points with masses m_1 and m_2, moving in elliptic Kepler orbits around one another. Unlike the classical elliptical three-body problem, we will assume that the orbital station is not a point mass, but a body of variable composition with a solid shell and a triaxial ellipsoid of inertia. We will assume that the mass of the orbital station m is exponentially variable: $m = m_0\exp(-\lambda t)$, $\lambda > 0$. We will also assume that as the working body becomes depleted, the position of the centre of mass of the orbital station, the directions of the principal axes of inertia with respect to its body, and also the values of the corresponding radii of inertia remain unchanged (which can be ensured, for example, by an appropriate change in the dimensions of the working body).

We will assume that a jet engine, which gives the orbital station a reaction acceleration **w** of constant modulus, which passes through its centre of mass, has an autonomous system for stabilizing the direction of the thrust in a system of coordinates $Oxyz$, which rotates together with the Moon. The origin of this system of coordinates is at the centre of mass of the Earth and the Moon, and the Ox and Oy axes lie in the plane of their orbits (the Ox axis is directed along the Earth–Moon line), while the Oz axis is perpendicular to this plane.

We will introduce two other rectangular systems of coordinates with origin at the centre of mass of the station: and orbital $CXYZ$ system (CZ axis is directed along the radius vector of the centre of mass C, drawn from the point O, the CX axis supplements the system up to the right one) and a system connected with the solid shell of the orbital station $Cx_1x_2x_3$ with axes directed along the principle axes of inertia of the orbital station. The position of the connected system of coordinates with respect to the orbital system is specified by the Euler angles ψ, φ, θ.

When calculating the force functions U_1 and U_2 of the Newtonian attraction forces applied to the orbital station due to the Earth and the Moon, we will assume that its characteristic dimension l is much less than the distance $r_i = [(x-a_i)^2 + y^2 + z^2]^{1/2}$ ($i = 1, 2$ and a_1 and a_2 are the coordinates of the Earth and the Moon in the $Oxyz$ system) between the orbital station and the Earth and the Moon. Then, neglecting terms of the order of $(l/a)^3$ and higher (a is the semiaxis of the Moon's orbit), we obtain the following approximate expressions for U_i ($i = 1, 2$) [6]

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$$U_i = \frac{\mu_i m}{r_i} - \frac{3\mu_i}{2r_i^3}\sum_{j=1}^{3}\left(A_j\gamma_{ij}^2 - \frac{A_j}{3}\right) \tag{1.1}$$

Here μ_1 and μ_2 are the gravitational parameters of the Earth and the Moon, respectively, A_j are the principal central moments of inertia of the orbital station, while

$$\gamma_{ij} = [(x - a_i)a_{1j} + ya_{2j} + za_{3j}]r_i^{-1} \tag{1.2}$$

are the direction cosines of the radius vectors r_i with respect to the connected axes $Cx_1x_2x_3$ where the quantities a_{sj} are expressed in terms of the Euler angles by the formulae

$$\begin{aligned}
&a_{11} = \cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta\\
&a_{12} = -\cos\psi\sin\varphi - \sin\psi\cos\varphi\cos\theta\\
&a_{13} = \sin\psi\sin\theta,\ a_{21} = \sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta\\
&a_{22} = -\sin\psi\sin\varphi - \cos\psi\cos\varphi\cos\theta,\ a_{23} = -\cos\psi\sin\theta\\
&a_{31} = \sin\varphi\sin\theta,\ a_{32} = \cos\varphi\sin\theta,\ a_{33} = \cos\theta
\end{aligned} \tag{1.3}$$

Taking into account the fact that the terms that occur in (1.1) which depend on the geometry of the masses are proportional to $(l/a)^2$ and are of the order of 10^{-14} when $l \leqslant 30$ m, we can ignore them in the equations of motion of the centre of mass, i.e. the translational motion of the orbital station can be considered separately from the rotational motion (but not vice versa).

Choosing the true anomaly of the Moon's orbit ν as the new independent variable and changing to Nechvil coordinates [2] ξ, η, ζ using the formulae (e is the eccentricity of the Moon's orbit)

$$x = \rho\xi,\ y = \rho\eta,\ z = \zeta,\ \rho = a(1 - e^2)/(1 - e\cos\nu)$$

and also taking into account the fact that the condition for the value of the reaction acceleration and its orientation in the rotating system to be fixed enables us to regard the force field as a potential field, we obtain the following equations of motion of the centre of mass of the orbital station (a dot denotes differentiation with respect to ν, and the quantity a is taken as the unit of length)

$$\begin{aligned}
&\ddot{\xi} - 2\dot{\eta} = -\psi(\nu)\frac{\partial W_1}{\partial \xi}\\
&\ddot{\eta} + 2\dot{\xi} = -\psi(\nu)\frac{\partial W_1}{\partial \eta}\\
&\ddot{\zeta} = \psi(\nu)\frac{\partial W_1}{\partial \zeta}
\end{aligned} \tag{1.4}$$

Here

$$W_1 = -\frac{\xi^2 + \eta^2}{2} + \frac{\zeta^2}{2}e\cos\nu - \left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) - \tilde{w}(1 - e\cos\nu)^2\left(\sigma_\xi\xi + \sigma_\eta\eta + \sigma_\zeta\zeta\right)$$

$$\mu = \frac{m_2}{m_1 + m_2},\quad \psi(\nu) = \frac{1}{1 - e\cos\nu},\quad \tilde{w} = \frac{wa^2\left(1 - e^2\right)^2}{\mu_1 + \mu_2}$$

$$\rho_1^2 = (\xi + \mu)^2 + \eta^2 + \zeta^2,\quad \rho_2^2 = (\xi + \mu - 1)^2 + \eta^2 + \zeta^2$$

and σ_ξ, σ_η, σ_ζ are the direction cosines of the reaction acceleration in the rotating system $O\xi\eta\zeta$.

The total potential energy, which determines the rotational motion of the orbital station, can be written as [6, 7]

$$\bar{W}_2 = \frac{1}{2}\dot{\nu}\sum_{s=1}^{3} A_s a_{3s}^2 + \frac{3}{2}\sum_{i=1}^{2} \frac{\mu_i}{\rho_i^3}\sum_{j=1}^{3}\left(A_j\gamma_{ij}^2 - \frac{A_j}{3}\right) \qquad (1.5)$$

Note that both system (1.4) and the potential energy (1.5) are analytic functions of the eccentricity of the Moon's orbit e, which can be regarded as a small parameter. When this parameter vanishes the system of equations of the translational-rotational motion of the orbital station becomes autonomous and possesses the generalized energy integral $T_2 + W_1 + \bar{W}_2 = \text{const}$ (T_2 is the quadratic part of the kinetic energy, expressed in terms of ξ, η, ζ, η, θ, ψ and their derivatives). When $e \neq 0$ this system of equations is periodic with period 2π with respect to the true anomaly ν.

2. We will show that when $e = 0$ the system of equations of translational-rotational motion of the orbital station has a family of steady motions, to which the positions of relative equilibrium of the orbital station in the rotating system of coordinates $O\xi\eta\zeta$ correspond. As follows from (1.3), the equilibrium values of the coordinates of the centre of mass of the orbital station (the new libration points) are found from the conditions which define the steady values of the function W_1 and which lead to the equations

$$\tilde{w}\sigma_\xi + \xi - \alpha_1(\xi+\mu) - \alpha_2(\xi+\mu-1) = 0$$

$$\tilde{w}\sigma_\eta + (1-\alpha)\eta = 0, \quad \tilde{w}\sigma_\zeta - \alpha\zeta = 0 \qquad (2.1)$$

$$\left(\alpha_1 = \frac{1-\mu}{\rho_1^3}, \quad \alpha_2 = \frac{\mu}{\rho_2^3}, \quad \alpha = \alpha_1 + \alpha_2\right)$$

From these equations one can either determine the coordinates of the libration points, given the value of the acceleration $\tilde{w}$ and the direction cosines σ_ξ, σ_η, σ_ζ, or, conversely, obtain the necessary value of the reaction acceleration and its orientation from the specified coordinates of the libration points. The first way is not very promising since it leads to the need to solve a complex system of non-linear equations in the coordinates $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ of the libration points. The inverse formulation of the problem enables one to obtain immediately the value of the necessary acceleration $\tilde{w}$ and its orientation.

In fact, by squaring each of Eqs (2.1) and adding, we obtain

$$\tilde{w}_2 = (\xi - \alpha_1(\xi+\mu) - \alpha_2(\xi+\mu-1))^2 + (1-\alpha)^2\eta^2 + \alpha^2\zeta^2 \qquad (2.2)$$

In a small neighbourhood of the classical libration point, the necessary values of $\tilde{w}$ will obviously be small and they can be calculated by expanding the right-hand side of (2.2) in a series in powers of the deviations

$$\xi_* = \bar{\xi} - \xi_0, \ \eta_* = \bar{\eta} - \eta_0, \ \zeta_* = \bar{\zeta} - \zeta_0$$

where ξ_0, η_0, ζ_0 are the coordinates of the classical libration points. For collinear libration points (which are the most interesting from the applied point of view) we will have (the zero subscript here and henceforth denotes that these quantities must be taken at the classical libration points)

$$\tilde{w}^2 = \left[\frac{\partial^2 W_1}{\partial \xi^2}\right]_0^2 \xi_*^2 + \left[\frac{\partial^2 W_1}{\partial \eta^2}\right]_0^2 \eta_*^2 + \left[\frac{\partial^2 W_1}{\partial \zeta^2}\right]_0^2 \zeta_*^2 + \ldots \qquad (2.3)$$

Hence, in the first approximation, for the same value of the acceleration $\tilde{w}$, we obtain an innumerable set of new libration points, situated on the surface of an ellipsoid, surrounding the classical libration point, with centre at this point. The necessary orientation of the reaction acceleration vector (its direction cosines) can be found from (2.1).

For the derivatives occurring in (2.3) we obtain

$$\left[\frac{\partial^2 W_1}{\partial \xi^2}\right]_0 = -(1+2\alpha_0),\ \left[\frac{\partial^2 W_1}{\partial \eta^2}\right] = \alpha_0 - 1,\ \left[\frac{\partial^2 W_1}{\partial \zeta^2}\right]_0 = \alpha_0$$

$$(\alpha_0 = (\alpha_1 + \alpha_2)_0)$$

Thus, for the classical libration point behind the Moon, denoting its distance from the Moon by ε ($\varepsilon \approx 0.156\ldots$), we will have

$$\alpha_0 = \mu(1+\varepsilon)^{-3} + (1-\mu)\varepsilon^{-3} \approx (1-\mu)\varepsilon^{-3}$$

Hence, for the values of the semiaxes of the ellipsoid a_ξ, a_η, a_ζ with centre at this point we obtain

$$a_\xi \approx \frac{1}{2}\tilde{w}\varepsilon^3,\ a_\eta \approx \tilde{w}\varepsilon^3,\ a_\zeta \approx \tilde{w}\varepsilon^3/(1-\mu)$$

Hence, the ellipsoid considered is (since $\mu << 1$) close to the ellipsoid of rotation, flattened along the $O\xi$ axis, with semiaxes along the two other coordinates axes, approximately double the semiaxis corresponding to the axis of rotation.

We will now find the positions of relative equilibrium of the orbital station itself as a solid in an orbital system of coordinates $CXYZ$. Taking into account the above assumption that the radii of inertia of the orbital station are constant we can regard the function $W_2 = \tilde{W}_2/m$ as the potential energy characterizing the rotational motion of the orbital station, instead of (1.5). This will not depend explicitly on time, since instead of the moments of inertia A_j the squares of the radii of inertia I_j^2 will occur in it, and these replace the corresponding moments of inertia in Euler's dynamic equations also. We will use the Euler angles ψ, φ, θ as the Lagrange coordinates, which define the rotational motion of the orbital station. The positions of relative equilibrium of the orbital station are then found from the system of equations

$$\frac{\partial W_2}{\partial \varphi} = 0,\ \frac{\partial W_2}{\partial \theta} = 0,\ \frac{\partial W_2}{\partial \psi} = 0 \tag{2.4}$$

In addition to the radii of inertia, these equations will contain, as parameters, the coordinates of the centre of mass, which are solutions of Eqs (2.1). Without considering the general case, in view of the complexity of the equations obtained, we will confine ourselves to the set of equilibrium positions with $\zeta = 0$. Turning to expression (1.5) and taking relations (1.2) and (1.3) into account, we can show that the first two equations of (2.4) are satisfied for any ξ and η with $\varphi = 0$ and $\theta = \pi/2$, and from the latter we obtain the equilibrium value of the angle $\psi = \psi_*$, where

$$\operatorname{tg}\psi^* = \frac{2\eta[\beta_1(\xi+\mu)+\beta_2(\xi+\mu-1)]}{\beta_1[(\xi+\mu)^2-\eta^2]+\beta_2[(\xi+\mu-1)^2-\eta^2]} \tag{2.5}$$

$$\beta_1 = (1-\mu)/\rho_1^5,\ \beta_2 = \mu/\rho_2^5$$

3. We will investigate the stability of the positions of relative equilibrium of the orbital station obtained. Since the phase coordinates, which define the rotational motion of the orbital station, do not occur in the equations of motion of the centre of mass, the stability of the positions of relative equilibrium of the centre of mass (the libration points) can be considered taking only Eqs (1.4) into account. Note that the potential energy W_1, when there is a reaction acceleration $\tilde{w}$, as can easily be shown, has no isolated minimum when $e = 0$ nor for any values of its variables and, consequently, we can only reckon on the possibility of gyroscopic stabilization of the libration points, i.e. only the necessary conditions of stability can be obtained. To do this we introduce the perturbations

$$y_1 = \xi - \bar{\xi},\ y_2 = \eta - \bar{\eta},\ y_3 = \zeta - \bar{\zeta}$$

and we set up the equations of perturbed motion, assuming $e = 0$ in (1.4). Omitting the non-linear terms, we obtain

$$
\begin{aligned}
&\ddot{y}_1 - 2\dot{y}_2 + b_{11}y_1 + b_{12}y_2 + b_{13}y_3 = 0 \\
&\ddot{y}_2 + 2\dot{y}_1 + b_{21}y_1 + b_{22}y_2 + b_{23}y_3 = 0 \\
&\ddot{y}_3 + b_{31}y_1 + b_{32}y_2 + b_{33}y_3 = 0
\end{aligned}
\tag{3.1}
$$

where

$$
b_{11} = -1 - \alpha_1\left[3\left(\frac{\bar{\xi}+\mu}{\rho_1}\right)^2 - 1\right] - \alpha_2\left[3\left(\frac{\bar{\xi}+\mu-1}{\rho_2}\right)^2 - 1\right]
$$

$$
b_{21} = b_{12} = -3\bar{\eta}\left[\beta_1(\bar{\xi}+\mu) + \beta_1(\bar{\xi}+\mu-1)\right],\; b_{31} = b_{13} = b_{21}\bar{\zeta}/\bar{\eta}
$$

$$
b_{22} = -1 + \alpha - 3\bar{\eta}^2\beta,\; b_{23} = b_{32} = -3\bar{\zeta}\bar{\eta}\beta,\; b_{33} = \alpha - 3\bar{\zeta}^2\beta
$$

$$
(\beta = \beta_1 + \beta_2)
$$

The corresponding values $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ of the reaction acceleration and its orientation can be found from the equilibrium equations (2.1). Since the acceleration itself does not occur in the expression for the coefficients of system (3.1), the stability domain is most easily constructed in the configuration space ξ, η, ζ, without using the equilibrium equations (2.1), which considerably simplifies the solution of the problem (here the expressions for the second partial derivative, which occur in the coefficients of system (3.1), are exactly the same as the corresponding expressions for the classical restricted three-body problem).

The necessary stability conditions of the trivial solution of system (3.1) consist of the requirement for all the roots of the characteristic equation

$$
\lambda^6 + b_4\lambda^4 + b_2\lambda^2 + b_0 = 0 \tag{3.2}
$$

$$
b_4 = 4 + b_{11} + b_{22} + b_{33}
$$

$$
b_2 = b_{11}b_{22} - b_{12}^2 - b_{31}^2 - b_{23}^2 + b_{22}b_{33} + b_{11}b_{33} + 4b_{33}
$$

$$
b_0 = b_{11}b_{22}b_{33} - b_{31}^2 b_{22} - b_{23}^2 b_{11} - b_{21}^2 b_{33} + 2b_{31}b_{12}b_{32}
$$

to be real and negative with respect to λ^2. This requirement can be satisfied by using the condition for the discriminant of the cubic equation corresponding to (3.2) (which guarantees that the roots are real) to be negative together with the Routh–Hurwitz conditions (which guarantee that they will be negative), written in this case as

$$
\frac{1}{4}\left(\frac{2}{27}b_4^3 - \frac{1}{3}b_2b_4 + b_0\right)^2 + \frac{1}{27^2}\left(3b_2 - b_4^2\right)^3 < 0 \tag{3.3}
$$

$$
b_4 > 0,\; b_0 > 0,\; b_4b_2 > b_0
$$

The set of values of the coordinates, which satisfy inequalities (3.3), define an innumerable set of new libration points that are stable in the first approximation. This set is bounded by a doubly connected surface. In Fig. 1 we show sections of this surface by the planes $\zeta = 0$ (symmetrical about the $\eta = 0$ axis of the doubly connected region, consisting of the unclosed ring and the crescent-shaped region, situated close to the collinear libration point L_3) and $\zeta = 0.27$ (which practically merge into a single line), obtained on a computer. The stability domain, therefore, splits into two parts: one of these consists almost entirely of points very distant from the Moon (☾) (at a distance or more from the Earth (⊕)), and the other (of somewhat smaller dimensions) is close to the outer collinear libration point L_3 and includes not only the collinear points but also points which lie on both sides of the $O\xi$ axis. The latter are particularly attractive for use for relay purposes, since they enable the orbital station to be seen simultaneously from the Moon and the Earth.

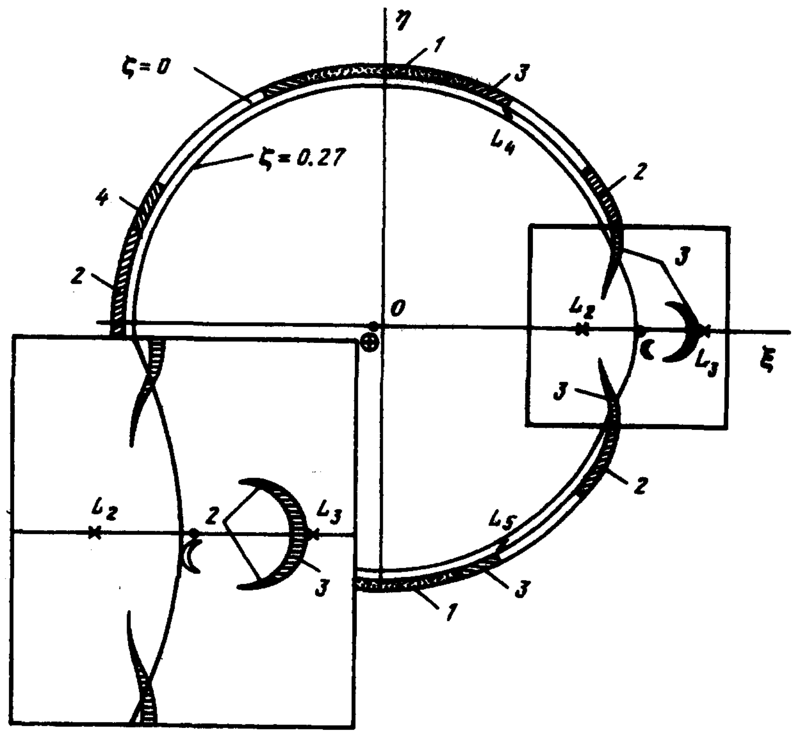

Fig. 1.

The maximum value of the necessary reaction acceleration in all stability domains is $\tilde{w} = 0.51$, for which $\tilde{w}_{\max}$ gives 1.39×10^{-3} m/s^2.

Note that, although the stability conditions obtained are only necessary for the system with potential forces considered (i.e. which can be represented in Hamilton form) there will also be the conditions for complete Birkhoff stability, with the exception of certain resonance sets [8].

We will now investigate the stability of the orientation of the orbital station with respect to the orbital system of coordinates, obtained in Section 2. Since we wish to obtain the sufficient stability conditions, we will set up the second derivatives of the potential energy W_2. Assuming $\varphi = 0$, $\theta = \pi/2$, $\psi = \psi_*$, we obtain

$$\frac{\partial^2 W_2}{\partial\varphi\partial\psi} = \frac{\partial^2 W_2}{\partial\theta\partial\psi} = \frac{\partial^2 W_2}{\partial\varphi\partial\theta} = 0$$

$$\frac{\partial^2 W_2}{\partial\psi^2} = 3(I_3 - I_1)f_\psi, \quad \frac{\partial^2 W_2}{\partial\varphi^2} = (I_2 - I_1)f_\varphi, \quad \frac{\partial^2 W_2}{\partial\theta^2} = (I_2 - I_3)f_\theta$$

$$f_\psi = \left[(h_1 - h_2)\cos 2\psi - 2h_{12}\sin 2\psi\right]_*$$

$$f_\varphi = \left[3((h_1\cos^2\psi + h_2\sin^2\psi) + h_{12}\sin 2\psi) - 1\right]_* \tag{3.4}$$

$$f_\theta = \left[3((h_1\sin^2\psi + h_2\cos^2\psi) - h_{12}\sin 2\psi) - 1\right]_*$$

$$h_1 = \beta_1(\xi + \mu)^2 + \beta_2(\xi + \mu - 1)^2, \quad h_2 = \beta\eta^2$$

$$h_{12} = \left[\beta_1(\xi + \mu) + \beta_2(\xi + \mu - 1)\right]\eta$$

The asterisk denotes the result of substituting, instead of ψ, the values determined by (2.5). It can be shown that after this substitution, for the set of libration points situated in the neighbourhood of the point L_3, where $\eta < \xi$, we will have $f_\psi > 0$. The functions f_φ, f_θ in this region, as computer calculations showed, can take both positive and negative values. Here only two of the four possible different cases are obtained

$$1)\, f_\psi > 0, f_\varphi > 0, f_\theta > 0 \text{ and } 2)\, f_\psi > 0, f_\varphi > 0, f_\theta < 0.$$

The sufficient conditions for stability (including the requirements that the second derivatives defined by (3.4) should be positive) will then be given by the following inequalities, which the radii of inertia of the orbital station must satisfy, respectively

$$1)\, I_2 > I_3 > I_1;\ 2)\, I_3 > I_2 > I_1.$$

Similar stability conditions can also be obtained for other values of the coordinates of the centre of mass, situated far from the Moon and of less practical interest. In Fig. 1, in the region of the stability of the centre of mass with $\zeta = 0$, we have indicated the regions of different combinations of signs of the functions f_ψ, f_φ and f_θ, in which sufficient conditions for stability can be satisfied directly by choosing a special geometry of the masses (in region 1: $f_\psi < 0, f_\varphi < 0, f_\theta > 0$, $(I_1 > I_2 > I_3)$; in region 2: $f_\psi > 0$, $f_\varphi > 0, f_\theta > 0$, $(I_2 > I_3 > I_1)$; in region 3: $f_\psi > 0, f_\varphi > 0, f_\theta > 0$, $(I_3 > I_2 > I_1)$; in region 4: $f_\psi < 0, f_\varphi > 0, f_\theta > 0$, $(I_2 > I_1 > I_3)$; the sufficient conditions are not satisfied in the unhatched part of the region).

The results obtained on the existence and stability of the positions of relative equilibrium of an orbital station also enable us to draw certain conclusions regarding the motion when the eccentricity e is non-zero and fairly small. Thus, when $e \neq 0$, both the system of equations (1.4) and the system of equations which define the rotational motion of the orbital station is 2π-periodic with respect to the true anomaly of the Moon ν (which plays the role of the independent variable) and is analytic with respect to the eccentricity e (it can play the role of a small parameter). Consequently, from Poincaré's theorem [9] in the neighbourhood of the equilibrium positions obtained there can be 2π-periodic motions only if the characteristic equation (3.2) has no roots of the form $\pm\, ki$ $(k = 0, 1, 2, \ldots)$. Since the system of equations of translational-rotational motion of the orbital station considered can be written in Hamilton form, it is reversible and, according to investigations carried out previously [10, 11], the characteristic exponents of the equations in variations for these periodic motions, apart from the square of the small parameter e, will be identical with the roots of characteristic equation (3.2). Hence, the stability domain for the libration points for $e = 0$, constructed above, will be practically identical for small $e \neq 0$ with the stability domain for this periodic motion, with the exception of a set of points which lead to parametric resonance, which arises [11] when $\lambda_s, \pm \lambda_j = ik$ where k is an integer.

Only in such cases can the stability of the periodic motions break down for as small a value of $e \neq 0$ as desired. Hence, with the exception of a set of points of infinitesimal measure, the stability domain constructed is also conserved in practice when the eccentricity of the lunar orbit is taken into account. If we eliminate the set of points of the domain corresponding to second-order and third-order resonances [12], then, as follows from the general theory of Hamiltonian systems, the stability conditions obtained also guarantee stability when non-linear terms up to the third order inclusive are taken into account in the equations of perturbed motion.

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